ON CONSTRUCTION AND ANALYSIS OF $p \times q$ CONFOUNDED ASYMMETRICAL FACTORIAL DESIGNS

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(Received : August, 1976)

INTRODUCTION

The theory behind the construction and analysis of symmetrical factorials has been developed considerably through the work of Fisher, Yates, Bose, Rao, Kishen, Das and several other workers. The development in this sphere has perhaps surpassed the need of the experimenter. However, this is not true in the case of asymmetrical factorials which are more useful and flexible to meet the demands of the subject-matter specialists. Starting with the introduction of confounded asymmetrical factorial designs by Yates[5], a number of methods of construction of such designs have since been given. Prominent among these methods are those due to Kishen and Srivastava[4] and Das[1]. But these methods are still not satisfactory, as in many cases suitable designs with reasonable number of replications are not available. For example, if there be an experiment involving two factors at 5 and 7 levels respectively, no design involving 2 or 3 replications seems possible through the existing methods of construction. Moreover there is no satisfactory method of analysis of such confounded designs because the usual method of analysis by dividing into components is not possible for such experiments involving two or more prime number of levels (other than 2). An attempt has thus been made to obtain such designs by linking the asymmetrical design with some corresponding symmetrical design.

METHOD

The main technique adopted for construction of $p \times q$ confounded asymmetrical factorial design is to convert the asymmetrical design into a symmetrical design of the series 2^n by suitably designating the levels of each of the p and q levelled factors of the asymmetrical design by one or more combinations of a certain number of factors each at 2 levels. Some precautions have to be taken while thus designating the levels of the two factors of the asymmetrical design so that a suitable working relationship between the main effects and interactions of the asymmetrical design on the one hand and the main effects and interactions of the 2^n series of design on the other can be established. The following is a description of the actual method. It has been discussed with reference to the 5×7 design though the method is quite general and can be extended easily. Let A_1 and A_2 be the two factors of the asymmetrical design at levels 5 and 7 respectively. Choose a group of n_1 factors corresponding to the levels of factor A_1 and n_2 factors corresponding to the levels of factor A_2 where each of n_1 and n_2 factors are at two levels say 0 and 1 such that 5 (the number of levels of A_1 factor) lies between 2^{n_1-1} and 2^{n_1} and similarly 7 lies between 2^{n_2-1} and 2^{n_2}. The values of each of n_1 and n_2 is 3 as $2^2<5<2^3$ and $2^2<7<2^3$. The 2^{n_1} or 2^{n_2}, that is 8 combinations are first divided into 4 pairs by confounding all the main effects and interactions of the first (n_1-1) or (n_2-1) *i.e.*, 2 of the factors. Let the 3 factors of the symmetric design corresponding to the 5 levels of the factor A_1 be denoted by X_{11}, X_{12} and X_{13}. Similarly let X_{21}, X_{22} and X_{23} denote the factors of the symmetric design corresponding to A_2 of the asymmetric design. The 8 treatment combinations grouped into 4 pairs give 4 blocks each of size 2. We arrange these blocks in such a way that each of the first 2 blocks contains the 0 level of the first factor *i.e.*, X_{11} or X_{21}. Consider the case of A_2 factor. In this case we separate out the first $7-2^{3-1}=3$ of the 4 blocks and use the different combinations in these blocks to designate $(2\times7-2^3)=6$ levels of the factor A_2. Thus the 6 combinations in the first 3 blocks are utilised to denote 6 levels of the factor A_2. Now levels of the factor A_2 which remain to be designated yet is only one and the number of blocks of the symmetric design left unused is also one. Hence we shall use each of the two combinations in the last block to designate the 7th level of A_2. Hence in case of A_2 with 7 levels, as the factors of the symmetric design being X_{21}, X_{22}, X_{23}, by confounding X_{21}, X_{22}, and $X_{21}X_{22}$ we have the following 4 blocks:

0 0 0	0 1 0	1 0 0	1 1 0
0 0 1	0 1 1	1 0 1	1 1 1

The 6 combinations in the first 3 blocks are used to designate the 6 levels of A_2 and each of the two combinations (110) and (111) in the last block is used to designate the 7th level of A_2 when the levels of A_2 are arranged in any suitable order. In case of A_1 with 5 levels, the pseudo factors in the symmetric design being X_{11}, X_{12} and X_{13}, then the levels of A_1 and the combinations of the corresponding factors X_{11}, X_{12} and X_{13} are as shown below:

Levels of A_1	*Combinations of* X_{11}	X_{12}	X_{13}
0	0	0	0
1	0	0	1
2	0	1	0
2	0	1	1
3	1	0	0
3	1	0	1
4	1	1	0
4	1	1	1

Through this method, the asymmetrical factorial 5×7 is converted into a symmetrical factorial of size 2^6. The level designation of the asymmetrical and symmetrical design will be as follows:

Asymmetrical Design *Levels of*		**Symmetrical Design** (*Corresponding to* A_1)			(*Corresponding to* A_2)		
A_1	A_2	X_{11}	X_{12}	X_{13}	X_{21}	X_{22}	X_{23}
0	0	0	0	0	0	0	0
1	1	0	0	1	0	0	1
2	2	0	1	0	0	1	0
2	3	0	1	1	0	1	1
3	4	1	0	0	1	0	0
3	5	1	0	1	1	0	1
4	6	1	1	0	1	1	0
4	6	1	1	1	1	1	1

This amounts to the fact that the following block of the symmetrical 2^6 factorial

0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	0
1	1	1	1	1	1

converts to the following asymmetrical 5×7

0	0
1	1
2	2
2	3
3	4
3	5
4	6
4	6

by following the nomenclature for designating the levels as explained earlier.

We shall now consider the linking of the contrasts of the main effects of the factors of the asymmetric design with suitable main effect and interaction contrasts of the pseudo factors in the symmetrical design.

Consider the case of $A_2=7$. In this case each of the 3 contrasts which were confounded while forming the 4 blocks, denotes a main effect contrast of the asymmetric factor. If in any contrast the two combinations in a block each of which defines the same level of asymmetric factor, have different signs, these two combinations get eliminated from the contrast of A_2 as both of them denote the same level. This will result in disturbing the orthogonality of the contrast. That is though the contrasts of main effects and interactions of the factors in the symmetric design are orthogonal the corresponding contrasts among the levels of A_2 need not be so. As such we shall define only those orthogonal contrasts of the 8 combinations of the symmetric design which are contrasts among some or all the 4 blocks or contrasts among those combinations only each of which defines a separate level of A_2. We have already got 3 orthogonal contrasts among the 4 block totals. For the remaining 3 contrasts of A_2 we shall proceed as follows. As we have used 6 combinations to denote a separate level of A_2 and as 6 is greater than 4, the total number of blocks, then we take that $\frac{1}{2}$ fraction of 8 combinations which contains 4 combinations which are used singly to denote a separate level of A_2 and get 2 contrasts by allotting opposite signs to the two combinations in each block. It will be recalled that we have written the 6 combinations each of which was used to designate a separate level of A_2 such that the first 4 combinations in the first two blocks begins with 0 so that this set gives a $\frac{1}{2}$ fraction of the 8 combinations of the symmetric design with the identity group $I = X_{21}$. We have got in all 5 main effect contrasts of A_2. For the last contrast *i.e.*, the 6th we take $\frac{1}{4}$th fraction of the combination which contains different levels of A_2 but not covered by the $\frac{1}{2}$ fraction already considered. We shall thus get a complete set of 6 orthogonal contrasts among the levels of the factors X_{21}, X_{22} and X_{23} and these represent the main effect contrasts of A_2.

In case of $A_1 = 5$, each of the 3 contrasts which were confounded while forming the blocks, represent a main effect contrast of A_1. For the remaining one contrast we consider $\frac{1}{4}$th of the treatment combinations each of which denoted a separate level of A_1. This one-fourth can evidently be made to come from a fraction with the identity group $I = X_{11} = X_{12} = X_{11}\ X_{12}$. We have thus set up a complete correspondence between the total number of treatment combinations of the symmetric design and the main effect contrasts of the factors of asymmetrical design. We shall now illustrate the method of linking the contrasts by considering the case of $A_2 = 7$.

Table I shows the levels of A_2, using codes 0, 1, 2 3, 4, 5 and 6 together with the codes in form of the combinations of three factors X_{21}, X_{22} and X_{23} each at levels 0 and 1.

The main effect contrasts of A_2 are shown from Cols. 3 to 8 of Table I. The contrasts are evidently orthogonal. There is one more contrast *viz.*, the difference between the last two combinations which is orthogonal to each of the six main effect contrast. This contrast will contribute towards error as both the combinations (110) and (111) denote the 7th level of A_2. Let us denote this contrast by $X_{23}(e_1)$.

In case of $A_1 = 5$, the main effects and interactions are given in Table II.

TABLE I

Main Effects and Interactions of X_{21}, X_{22} and X_{23}

1	2			3	4	5	6	7	8
A_2	X_{21}	X_{22}	X_{23}	X_{21}	X_{22}	X_{21} X_{22}	X_{23} $=X_{21}X_{23}$	X_{22} $X_{23}=$ X_{21} X_{22} X_{23}	$X_{23}=X_{21}$ $X_{23}=$ X_{22} $X_{23}=$ $X_{21}X_{22}X_{23}$
0	0	0	0	+	+	+	+	+	0
1	0	0	1	+	+	+	—	—	0
2	0	1	0	+	—	—	+	—	0
3	0	1	1	+	—	—	—	+	0
4	1	0	0	—	+	—	0	0	+
5	1	0	1	—	—	—	0	0	—
6	1	1	0	—	+	+	0	0	0
6	1	1	1	—	—	+	0	0	0

TABLE II

Main Effects and Interactions of X_{11}, X_{12} and X_{13}

1	2			3	4	5	6
A_1	$X_{11}X_{12}X_{13}$			X_{11}	X_{12}	$X_{11}X_{12}$	$X_{13}=$ $X_{11}X_{13}=$ $X_{12}X_{13}=$ $X_{11}X_{12}X_{13}=$
0	0	0	0	+	+	+	+
1	0	0	1	+	+	+	—
2	0	1	0	+	—	—	0
2	0	1	1	+	—	—	0
3	1	0	0	—	+	—	0
3	1	0	1	—	+	—	0
4	1	1	0	—	—	+	0
4	1	1	1	—	—	+	0

In this case the error degrees of freedom will evidently be 3. Let the error component for the two combinations (010) and (011) be denoted by $X_{13}(e_1)$, for the two combinations (100) and (101) by $X_{13}(e_2)$ and for the two combinations (110) and (111) by $X_{13}(e_3)$.

INTERACTION OF ASYMMETRIC DESIGN

We have so far expressed contrasts representing main effects of the factors A_1 and A_2 in the asymmetric design in terms of main effect and interaction contrasts of factors X_{11}, X_{12}, X_{13} and X_{21}, X_{22}, X_{23} forming a symmetrical design and shall now consider the interactions among A_1 and A_2. Nine of the total of 24 degrees of freedom of A_1A_2 which are of the type XY where X is any main effect or interaction of the factors X_{11} and X_{12} and Y is similarly any main effect or interaction of the factors X_{21} and X_{22} are obtainable from the following 9 interactions :

$X_{11}X_{21}$, $X_{11}X_{22}$, $X_{11}X_{21}X_{22}$, $X_{12}X_{21}$, $X_{12}X_{22}$, $X_{12}X_{21}X_{22}$, $X_{11}X_{12}X_{21}$, $X_{11}X_{12}X_{22}$ and $X_{11}X_{12}X_{21}X_{22}$ and each of these interactions is obtained by using all the 64 combinations of 2^6 factorial.

The following 6 components of A_1A_2 which will be of the type XY', where Y' is any interaction (or main effect) of the factors X_{21}, X_{22} and X_{23} but involving the last factor X_{23} or the main effect X_{23}, are to be obtained from $\frac{1}{2}$ fraction of 2^6, defined by identity group

$$I=X_{21}.$$

$I=X_{21}$; $X_{11}X_{23}$, $X_{11}X_{22}X_{23}$, $X_{12}X_{23}$, $X_{12}X_{22}X_{23}$, $X_{11}X_{12}X_{23}$ and $X_{11}X_{12}X_{22}X_{23}$.

There are three more interactions of the type XY' which have to be obtained from the $\frac{1}{4}$th fraction defined by $I=X_{21}=X_{22}=X_{21}X_{22}$ involving combinations (100) and (101) of the factors X_{21}, X_{22} and X_{23} as indicated in Col. 8 of Table I. These components are $I=X_{21}=X_{22}=X_{21}X_{22}$; $X_{11}X_{23}$, $X_{12}X_{23}$ and $X_{11}X_{12}X_{23}$. Three of the remaining six components belong to the type $X'Y$ which have to be obtained from $\frac{1}{4}$th fraction of 2^6 defined by $I=X_{11}=X_{12}=X_{11}X_{12}$ and involving combinations (000) and (001) of the factors X_{11}, X_{12} and X_{13} as indicated in Col. 6 of Table II. These components are given by $X_{13}X_{21}$, $X_{13}X_{22}$ and $X_{13}X_{21}X_{22}$ with identity group $I=X_{11}=X_{12}=X_{11}X_{12}$.

Two of the remaining three components are of the type $X'Y'$ which have to be obtained from 1/8th fraction of 2^6 defined by

$$I=X_{11}=X_{12}=X_{11}X_{12}=X_{21}=X_{11}X_{21}=X_{12}X_{21}=X_{11}X_{12}X_{21}$$

involving combinations (000) and (001) of X_{11}, X_{12} and X_{13} among all those combinations which have level 0 of X_{21}. These components are $X_{13}X_{23}$ and $X_{13}X_{22}X_{23}$ with identity group as indicated above. The last interaction component is given by aliases of $X_{13}X_{23}$ obtainable from the identity group $I=X_{11}=X_{12}=X_{21}=X_{22}$ and their generalised interactions. Actually this contrast is obtained from the four combinations involving (000), (001) of X_{11}, X_{12} and X_{13} and (100), (101) of X_{21}, X_{22} and X_{23}.

We are now in a position to give a complete break-up of the total degrees of freedom of the 5×7 design of the factors A_1, A_2 as below :

	d.f.	
A_1	4	Corresponding to the contrasts of X_{11}, X_{12} and X_{13}.
A_2	6	Corresponding to the contrasts of X_{21}, X_{22} and X_{23}.
A_1A_2	24	Corresponding to the contrats of interaction involving factors from both the groups as indicated above.
Error	29	As discussed below.
Total	63	

As indicated earlier there are 3 *d.f.* as error among the 7 contrasts of X_{11}, X_{12} and X_{13} representing A_1 at 5 levels. Similarly there is 1 *d.f.* as error among the seven contrasts of the factors X_{21}, X_{22} and X_{23} of A_2. As indicated earlier denoting these contrasts by $X_{13}(e_1)$, $X_{13}(e_2)$ and $X_{13}(e_3)$ and $X_{23}(e_1)$ for A_1 and A_2 respectively we shall get the following total error *d.f.*

	d. f.	
$X_{13}(e_i)$	$1\times 3=3$	$(i=1, 2, 3)$
$X_{23}(e_1)$	1	
$A_2X_{13}(e_i)$	$6\times 3=18$	$(i=1, 2, 3)$
$A_1X_{23}(e_1)$	$4\times 1=4$	
$X_{13}(e_i)\,X_{23}(e_1)$	$3\times 1=3$	$(i=1, 2, 3)$
	29	

Although we have established a complete correspondence of the degrees of freedom of the two designs, it is not essential to obtain the analysis of the asymmetrical design by exploiting the above correspondence. Actually the various main effects and interactions sum of squares can be obtained by forming two-way tables. But when blocking is involved complication comes and it becomes necessary to exploit the correspondence for adjusting the affected components.

CONSEQUENCE OF BLOCKING

Though blocking does not involve any difficulty for the construction of the design, it necessitates adjustment while obtaining some of the interaction or main effect contrasts of the 5×7 design. It is not necessary that the effects, corresponding to the factors X_{11}, X_{12}, X_{13} for A_1 and X_{21}, X_{22}, X_{23} for A_2, which will be confounded in 2^6 factorial remain confounded in the 5×7 design. As such it becomes necessary to isolate the components of A_1A_2 which get confounded in the 5×7 design. Once this is done the sum of squares due to each degree of freedom of A_1A_2 which is confounded is obtained by expressing the contrast through the sign table as presented in Tables I and II. All these sum of squares are added up to get the total sum of squares due to the confounded components of A_1A_2. The total sum of squares due to A_1A_2 is obtained from the two-way table corresponding to the factors A_1 and A_2 ignoring blocking and from this we subtract the sum of squares due to the confounded components of A_1A_2 to get the corrected sum of squares due to A_1A_2.

In this case while blocking some of the error components also get mixed up with the blocks. As such they also have to be separated out from the error S.S. in order to get the corrected error sum of squares. The error contrasts which are confounded can be detected in the same manner as for detecting the interaction components *i.e.* by forming the sign table.

REMARKS

The present method of construction of confounded asymmetrical design bears resemblance with some of the existing methods of construction of such designs. One is due to Das and Rao[3] and the other due to Das[1]. Das and Rao constructed the series $3^n\times2^m$ by establishing its correspondence with a symmetrical design of the series 2^p. The main difference between the approach of Das[1] and the present approach is that while Das used a fraction of a suitable symmetrical design for constructing an asymmetrical design which required more than one replication for balance, the present technique uses the whole of the symmetrical design by utilizing part of the contrasts of the symmetrical design to obtain the contrasts of the asymmetrical design while the remaining contrasts are made to provide error contrasts and thus no further replication need be taken. The analysis of the asymmetrical design proceeds in the usual way by forming two-way table etc. The linking of the asymmetrical design with the symmetrical design is used to separate out the affected interactions as well as the affected error components and is also utilised for adjusting the sum of squares of the interaction and error components.

Another point worth mentioning is that as the confounded interactions and error components depend upon the initial set of interactions that are confounded in the symmetrical design, in the practical situation, the experimenter may have to face a laborious trial and error method for the optimum set. To overcome this difficulty we give below a rule by which the number of confounded interactions as well as confounded error components can be kept to the minimum. By following the method of construction of confounded symmetrical factorial designs given by Das[2] we have to write the pseudo factors corresponding to A_1 first and then those corresponding to A_2. Next a unit matrix is written below the factors of A_1. Next while filling up the subsequent columns we have to keep the elements in the last row zero excepting the last column and no two such added columns should be identical.

SUMMARY

Method of construction and analysis of a $p \times q$ confounded asymmetrical factorial design, based on confounded symmetrical factorial designs with factors each at two levels, has been described. The technique is illustrated through a 5×7 asymmetrical factorial design. The technique can easily be extended to construct confounded asymmetrical factorial designs, involving any type of levels.

ACKNOWLEDGMENT

The author is deeply indebted to Dr. D. Singh, Director, Institute of Agricultural Research Statistics, for the keen interest evinced by him in the preparation of this paper.

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